The CHSH Inequality: Classical Limits vs Quantum Entanglement

The setting

Two distant observers:

- Alice measures one of two observables: Q or R
- Bob measures one of two observables: S or T
- Each measurement outcome is ± 1

Question:

Can quantum correlations exceed all classical correlations?

Einstein's Objection and the EPR Paradox (1935)

Einstein, Podolsky and Rosen (EPR) argued that quantum mechanics might be **incomplete**.

They proposed three classical principles:

- **Realism:** Physical properties exist prior to measurement
- **Locality:** No instantaneous influences faster than light
- **Freedom (measurement independence):** Experimenters can freely choose settings

Quantum entanglement predicts strong correlations between distant systems:

"Spooky action at a distance" — Einstein

Bell (1964): Any theory satisfying realism + locality obeys certain inequalities.

CHSH inequality: A testable version using two observers and two measurements each.

Local Hidden Variable Model

Assume a hidden variable λ with distribution $\rho(\lambda)$.

Measurement outcomes are predetermined functions:

$$Q(\lambda), R(\lambda), S(\lambda), T(\lambda) \in \{-1, +1\}$$

Correlation function:

$$E(Q, S) = \int d\lambda \, \rho(\lambda) \, Q(\lambda) S(\lambda)$$

This captures all classical correlations consistent with locality.

Measurement Independence ("Freedom of Choice")

In Bell/CHSH tests, experimenters choose measurement settings:

$$x \in \{Q, R\} \quad \text{(Alice)}, \qquad y \in \{S, T\} \quad \text{(Bob)}.$$

Measurement independence assumption:

The choices of settings are statistically independent of the hidden variables.

$$P(x, y \mid \lambda) = P(x, y) \quad \Longleftrightarrow \quad P(\lambda \mid x, y) = P(\lambda).$$

Interpretation:

- Experimenters can freely choose measurement settings
- Hidden variables cannot "predict" or influence these choices
- No pre-established correlation between settings and system state

Why Measurement Independence Matters

If measurement independence fails, local hidden-variable models could reproduce quantum correlations.

Superdeterministic scenario:

- Hidden variables λ determine both
  - outcomes of measurements
  - experimenters' choices of settings
- Apparent Bell violation becomes classically explainable

Physical interpretation:

- Would require extremely strong pre-existing correlations across spacetime
- Often viewed as conspiratorial or implausible
- But logically consistent with locality

Experimental Safeguards

Modern Bell tests attempt to enforce independence by choosing settings randomly:

- Fast quantum random-number generators
- Space-like separation between choice and measurement
- Even astronomical sources (starlight, quasars) used as randomness

Goal: ensure settings are not causally connected to hidden variables.

Conclusion:
Bell violations rule out local realism *provided measurement independence holds.*

The CHSH Expression

Define

$$S_{\text{CHSH}} = E(Q,S) + E(Q,T) + E(R,S) - E(R,T)$$

Goal:

- Determine the maximum value allowed classically
- Compare with quantum predictions

Key Step: Deterministic Bound

For a fixed hidden variable value λ define

$$X_\lambda = QS + QT + RS - RT$$

Factor:

$$X_\lambda = Q(S + T) + R(S - T)$$

Bounding X_λ

Since $S, T = \pm 1$:

$$S + T \in \{-2, 0, 2\}, \quad S - T \in \{-2, 0, 2\}$$

Exactly one of these is nonzero.

Therefore:

$$|X_\lambda| = 2$$

Averaging over λ gives:

$$\boxed{|S_{\text{CHSH}}| \leq 2}$$

This is the classical CHSH inequality.

Pauli Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For a unit vector $\hat{n} = (n_x, n_y, n_z)$:

$$\hat{n} \cdot \vec{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$$

Represents measurement of spin along direction $\hat{n}$.
Eigenvalues: ± 1.

Quantum State

Use the singlet Bell state:

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Key identity:

$$\boxed{\langle\Psi^-|\sigma_i \otimes \sigma_j|\Psi^-\rangle = -\delta_{ij}}$$

Identical spin components are perfectly anti-correlated.

General Correlation Formula

Let

$$Q = \hat{q} \cdot \vec{\sigma}, \quad S = \hat{s} \cdot \vec{\sigma}$$

Then

$$E(Q,S) = \langle \Psi^- | (\hat{q} \cdot \vec{\sigma} \otimes \hat{s} \cdot \vec{\sigma}) | \Psi^- \rangle$$

Expand:

$$= \sum_{i,j} q_i s_j \langle \Psi^- | \sigma_i \otimes \sigma_j | \Psi^- \rangle$$

Using the identity:

$$\boxed{E(Q,S) = -\hat{q} \cdot \hat{s}}$$

Geometric Interpretation

$$E(Q,S) = -\cos\theta_{qs}$$

where θ_{qs} is the angle between the measurement directions.

Thus correlations depend purely on geometry of the Bloch sphere.

Choice of Alice's Observables

Choose orthogonal directions:

$$Q = \sigma_z, \quad R = \sigma_x$$

Corresponds to unit vectors:

$$\hat{q} = \hat{z}, \quad \hat{r} = \hat{x}$$

Angle between them: 90°.

Choice of Bob's Observables

Choose bisectors of Alice's directions:

$$S = \frac{\sigma_z + \sigma_x}{\sqrt{2}}, \quad T = \frac{\sigma_z - \sigma_x}{\sqrt{2}}$$

Unit vectors:

$$\hat{s} = \frac{\hat{z} + \hat{x}}{\sqrt{2}}, \quad \hat{t} = \frac{\hat{z} - \hat{x}}{\sqrt{2}}$$

Each makes a 45° angle with both $\hat{z}$ and $\hat{x}$.

Compute Dot Products

$$\hat{q} \cdot \hat{s} = \frac{1}{\sqrt{2}}, \quad \hat{q} \cdot \hat{t} = \frac{1}{\sqrt{2}}$$

$$\hat{r} \cdot \hat{s} = \frac{1}{\sqrt{2}}, \quad \hat{r} \cdot \hat{t} = -\frac{1}{\sqrt{2}}$$

Quantum Correlations

Using $E = -\hat{a} \cdot \hat{b}$:

$$E(Q,S) = -\frac{1}{\sqrt{2}}, \quad E(Q,T) = -\frac{1}{\sqrt{2}}$$

$$E(R,S) = -\frac{1}{\sqrt{2}}, \quad E(R,T) = +\frac{1}{\sqrt{2}}$$

CHSH Value

$$S_{\text{CHSH}} = E(Q,S) + E(Q,T) + E(R,S) - E(R,T)$$

Substitute:

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{4}{\sqrt{2}}$$

$$\boxed{|S_{\text{CHSH}}| = 2\sqrt{2}}$$

Comparison with Classical Bound

- Classical local hidden variables: $|S| \leq 2$
- Quantum mechanics: $|S| \leq 2\sqrt{2}$

This is the **Tsirelson bound**.

Quantum entanglement produces correlations impossible in any classical local theory.

Why These Angles Are Optimal

- Alice's observables are orthogonal
- Bob's are the bisectors
- Maximizes constructive interference of three terms
- Minimizes the fourth

Geometrically:

$$\text{Orthogonal directions} \Rightarrow \text{maximum CHSH violation}$$

Does CHSH Miss "Nonlinear" Classical Correlations?

A common doubt: *CHSH is a linear inequality. Could there be "nonlinear" classical correlations that are stronger?*

Answer: No. CHSH already characterizes the *full power* of **all local classical models**.

In the Bell–CHSH scenario, a **classical local model** is:

$$P(a, b \mid x, y) = \int d\lambda\, \rho(\lambda)\, P(a \mid x, \lambda)\, P(b \mid y, \lambda),$$

where $x \in \{Q, R\}$ and $y \in \{S, T\}$, and $a, b \in \{\pm 1\}$.

This allows **arbitrarily complicated / nonlinear** dependence on λ inside $P(a \mid x, \lambda)$ and $P(b \mid y, \lambda)$.

Why It Suffices to Consider Deterministic Strategies

Even if $P(a \mid x, \lambda)$ and $P(b \mid y, \lambda)$ are probabilistic, the model is a **convex mixture** of deterministic response functions.

For each λ we may assume:

$$Q(\lambda), R(\lambda), S(\lambda), T(\lambda) \in \{\pm 1\}.$$

Reason: Any local stochastic strategy can be written as a mixture of deterministic ones (by absorbing extra randomness into λ).

So to maximize any Bell expression over classical local models, it suffices to check **deterministic** assignments.

Where Linearity Comes From: Convex Geometry

The set of all local classical correlations forms a **convex polytope** (the *local polytope*).

- **Vertices:** deterministic assignments $(Q, R, S, T) \in \{\pm 1\}^4$
- **All classical models:** convex combinations of vertices
- **Facets:** *linear* inequalities (hyperplanes) defining the boundary

Thus CHSH being *linear* is not a restriction on physics; it is a consequence of **convexity**: facets of a polytope are linear.

No "Nonlinear" Classical Trick Can Beat 2

For any deterministic assignment:

$$X_\lambda = QS + QT + RS - RT = Q(S + T) + R(S - T).$$

Since $S, T \in \{\pm 1\}$, exactly one of $(S + T)$ and $(S - T)$ equals ± 2 and the other is 0.

Therefore

$$|X_\lambda| = 2 \quad \text{for every } \lambda.$$

Averaging over λ:

$$\boxed{|S_{\text{CHSH}}| \leq 2.}$$

This bound already includes all classical (even highly nonlinear) local strategies.

Why Higher-Order "Nonlinear" Statistics Don't Help Here

In CHSH, outcomes are ± 1.

So for any outcome $a \in \{\pm 1\}$:

$$a^2 = 1, \quad a^3 = a, \quad a^4 = 1, \ \ldots$$

Any nonlinear function of outcomes reduces to a linear one. For example:

$$\mathbb{E}[a^2 b] = \mathbb{E}[b], \qquad \mathbb{E}[(ab)^2] = 1.$$

So the relevant correlators are exactly the **linear** expectations $\mathbb{E}[ab]$.

What Actually Beats CHSH: Quantum (and Beyond-Quantum) Correlations

Quantum violation is not from “nonlinearity” but from **incompatibility** (non-commutation):

$$[Q, R] \neq 0, \qquad [S, T] \neq 0.$$

So outcomes for Q and R cannot be simultaneously predetermined.

Correlation hierarchy:

$$\text{Classical Local} \subset \text{Quantum} \subset \text{No-signalling}.$$

Maximal CHSH values:

$$2 \quad < \quad 2\sqrt{2} \quad < \quad 4.$$

What CHSH Experiments Do — and Do Not — Establish

Bell/CHSH violations demonstrate that **local realistic theories** cannot explain observed correlations.

Specifically, experiments rule out the joint validity of:

- **Realism:** Outcomes reflect pre-existing properties
- **Locality:** No faster-than-light influence
- **Measurement independence:** Settings are freely chosen

However, the results do *not* uniquely confirm quantum mechanics as the only possible theory.

Alternative explanations remain logically possible:

- Nonlocal realist theories (e.g., hidden-variable models with instantaneous influences)
- Interpretations denying pre-existing properties (measurement-dependent reality)
- Theories violating measurement independence ("superdeterminism")
- Future theories reproducing the same correlations

Conclusion

- Pauli matrices allow explicit calculation of correlations
- Entangled states produce geometric correlations
- CHSH inequality shows quantum correlations exceed classical ones
- Verified experimentally many times

Quantum entanglement is stronger than any classical correlation