Contents

Quantum Computation and Quantum Information Theory (QCQIT) (Spring 2026)

Lecture Notes

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1 Lecture#1 (Jan 20)

[**Linear Algebra Review**]

[**Vector Spaces**] Throughout, all vector spaces are over the complex field $\mathbb{C}$

[**Notation.**] Vectors in a vector space V are denoted using Dirac notation, e.g., $|v\rangle \in V$.

[**Inner Product**] An inner product on a vector space V is a map $(\cdot,\cdot) : V \times V \to \mathbb{C}$ that satisfies the following properties:

1. $(|v\rangle, \sum_{i=1}^n c_i |w_i\rangle) = \sum_{i=1}^n c_i(|v\rangle, |w_i\rangle)$, where $c_i \in \mathbb{C}$
2. $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$
3. $(|v\rangle, |v\rangle) \geq 0$ (it is equal to 0 if and only if $|v\rangle = 0$)

We also write the inner product as $\langle v|w\rangle$. A inner product space is a vector space equipped with an inner product. The norm of a vector $|v\rangle \in V$ is defined by $|| \, |v\rangle \, || = \sqrt{\langle v|v\rangle}$.

[**Exc**] Show that $(\sum_{i=1}^n c_i |v_i\rangle, |w\rangle) = \sum_{i=1}^n c_i^*(|v_i\rangle, |w\rangle)$

[**Linear Operator**] For vector spaces V, W, a map/operator $A : V \to W$ is called linear if

$$A(\alpha |v\rangle + \beta |w\rangle) = \alpha A(|v\rangle) + \beta A(|w\rangle)$$

for all $\alpha, \beta \in \mathbb{C}$ and $|v\rangle, |w\rangle \in V$.

[**Linar Operator vs Matrix Representation**] Let V and W be vector spaces with ordered bases $\{|v_i\rangle\}_{i=1}^n$ and $\{|w_j\rangle\}_{j=1}^m$, respectively. Let A be an $m \times n$ matrix with entries in $\mathbb{C}$. Define a map $T_A : V \to W$ as follows: for $|v\rangle \in V$ written as $|v\rangle = \sum_{i=1}^n \alpha_i |v_i\rangle$, set $A|v\rangle = \sum_{j=1}^m \beta_j |w_j\rangle$ where the coefficients β_j are given by

$$\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} = A \cdot \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Check that T_A is linear.

Conversely, given any linear operator $T : V \to W$, there exists an $m \times n$ matrix A over $\mathcal{C}$ (depending on the chosen bases of V and W) such that $T = T_A$.

[**The Pauli Matrices**] Four particularly important matrices that describe linear operators $A : \mathbb{C}^2 \to \mathbb{C}^2$ are the Pauli matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

[**Dual:** $\langle v|$] For any vector $|v\rangle \in V$, define the map $\langle v| : V \to \mathbb{C}$ as follows: for any $|w\rangle \in V$,

$$\langle v| (|w\rangle) = \langle v|w\rangle$$

Check that $\langle v|$ is a linear operator. Suppose $\{|i\rangle\}$ be an orthonormal basis of V. Then, the matrix representation of $\langle v|$ is given by: let $|v\rangle = \sum_i \alpha_i |i\rangle$. Take any $|w\rangle = \sum_j \beta_j |j\rangle \in V$. Then

$$\langle v| (|w\rangle) = \langle v|w\rangle = \begin{bmatrix} \alpha_1^*, \dots, \alpha_n^* \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

Indeed,

$$\begin{aligned} \langle v| (|w\rangle) &= \langle v|w\rangle \\ &= (\sum_i \alpha_i |i\rangle \mid \sum_j \beta_j |j\rangle) \\ &= \sum_{i,j} \alpha_i^* \beta_j (|i\rangle, |j\rangle) \\ &= \sum_i \alpha_i^* \beta_i \end{aligned}$$

[**Outer Product**] For $|v\rangle \in V$, $|w\rangle \in W$, define the *outer product* maps $|w\rangle\langle v| : V \to W$ as follows: for any $|v\rangle' \in V$

$$|w\rangle\langle v|(|v\rangle') = |w\rangle \langle v|v'\rangle$$

Claim Outer product is linear.

[**Completeness Result**] Let $\{|i\rangle\}$ be an orthonormal basis of V. Then $\sum_i |i\rangle\langle i| = I_V$.

Proof: Take any $|v\rangle \in V$. Let $v = \sum_i \alpha_i |i\rangle$. Then

$$\begin{aligned} (\sum_i |i\rangle\langle i|)(|v\rangle) &= \sum_i (|i\rangle\langle i|)(|v\rangle) \\ &= \sum_i |i\rangle \langle i|v\rangle \\ &= \sum_i |i\rangle \alpha_i \\ &= |v\rangle \end{aligned}$$

[**Fact**] Let V and W be inner product spaces with $\{|v_i\rangle\}$ and $\{|w_j\rangle\}$ as orthonormal bases respectively. Let $A : V \to W$ be any linear operator. Then

$$A = \sum_{i,j} \langle w_j|A|v_i\rangle \; |w_j\rangle\langle v_i|$$

where $\langle w_j|A|v_i\rangle = (|w_j\rangle, A|v_i\rangle) \in \mathbb{C}$.

Proof: We first write,

$$
\begin{aligned}
A &= I_W A I_V \\
&= (\sum_j |w_i\rangle\langle w_j|)A(\sum_i |v_i\rangle\langle v_i|) \\
&= \sum_{i,j}(|w_j\rangle\langle w_j|)A(|v_i\rangle\langle v_i|)
\end{aligned}
$$

For a fix i, j and for any $|v\rangle \in V$, we have

$$
\begin{aligned}
(|w_j\rangle\langle w_j|)A(|v_i\rangle\langle v_i|)(|v\rangle) &= (|w_j\rangle\langle w_j|)A\,|v_i\rangle\,\langle v_i|v\rangle \\
&= (|w_j\rangle\langle w_j|)\langle v_i|v\rangle A\,|v_i\rangle \\
&= \langle v_i|v\rangle(|w_j\rangle\langle w_j|)A\,|v_i\rangle \\
&= \langle v_i|v\rangle\,|w_j\rangle\,\langle w_j|A|v_i\rangle \\
&= \langle w_j|A|v_i\rangle\langle v_i|v\rangle\,|w_j\rangle \\
&= \langle w_j|A|v_i\rangle|w_j\rangle\langle v_i|(|v\rangle)
\end{aligned}
$$

Therefore, as a map, $|w_j\rangle\langle w_j|)A(|v_i\rangle\langle v_i| = \langle w_j|A|v_i\rangle|w_j\rangle\langle v_i|$. Hence,

$$
A = \sum_{i,j}(|w_j\rangle\langle w_j|)A(|v_i\rangle\langle v_i|) = \sum_{i,j}\langle w_j|A|v_i\rangle|w_j\rangle\langle v_i|
$$

2 Lecture#2 (Jan 23)

[**Exc**] Consider the linear operator $A_Z : \mathbb{C}^2 \to \mathbb{C}^2$ defined by the matrix $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Let $\{|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ be the standard basis. Show that

$$
A_Z = |0\rangle\langle 0| - |1\rangle\langle 1|
$$

[**The Cauchy-Schwarz Inequality**] For any vectors $|v\rangle, |w\rangle \in V$,

$$
|\langle v|w\rangle|^2 \leq \langle v|v\rangle\langle w|w\rangle
$$

Proof:

[**Adjoint**] Consider the inner product spaces V and W. For any linear operator $A : V \to W$, the adjoint of A, denoted by $A^\dagger$, is the unique oeprator $A^\dagger : W \to V$ such that

$$
(|v\rangle, A\,|w\rangle) = (A^\dagger\,|v\rangle, |w\rangle), \text{ for all } |v\rangle \in V, \text{ and } |w\rangle \in W
$$

[**Exc**] Show that $A^\dagger$ is unique and also it is linear.

[**Exc**] $(A\,|v\rangle, |w\rangle) = (|v\rangle, A^\dagger\,|w\rangle)$, for all $|v\rangle \in V$ and $|w\rangle \in V$

[**Exc**] $(A^\dagger)^\dagger = A$

[**Exc**] For linear operators A, B, show that $(AB)^\dagger = B^\dagger A^\dagger$.

[**Dual**] For any $|v\rangle \in V$, consider the linear operator $S_{|v\rangle} : \mathbb{C} \to V$ defined as follows: for any $\alpha \in \mathbb{C}$, $S_{|v\rangle}(\alpha) = \alpha\,|v\rangle$. Then

$$
(S_{|v\rangle})^\dagger = \langle v|,
$$

where $\langle v| : V \to \mathbb{C}$ is the dual linear operator corresponding to $|v\rangle$. We drop the notation $S_{|v\rangle}$ and using $|v\rangle$ to think of it as the corresponding operator and therefore write

$$|v\rangle^\dagger = \langle v|$$

.

Proof: Take any $\alpha \in \mathbb{C}$ and $|u\rangle \in V$.

$$\begin{aligned}
(S_{|v\rangle}(\alpha), |u\rangle) &= (\alpha\, |v\rangle, |u\rangle) \\
&= (\alpha, \langle v|u\rangle) \\
&= (\alpha, \langle v|\,(|u\rangle))
\end{aligned}$$

Therefore, $(S_{|v\rangle})^\dagger = \langle v|$, i.e., $|v\rangle^\dagger = \langle v|$.

[**Exc**] For any linear operator A and $|v\rangle \in V$, we have $(A\,|v\rangle)^\dagger = \langle v|\, A^\dagger$

[**Fact**] For any $|v\rangle, |w\rangle \in V$, $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$

Proof: Take any $|v_1\rangle \in V$ and $|w_1\rangle \in W$. Then

$$\begin{aligned}
(|w\rangle\langle v|(|v\rangle_1), |w_1\rangle) &= (|w\rangle\, \langle v|v_1\rangle, |w_1\rangle\rangle) \\
&= (\langle v|v_1\rangle, |w\rangle^\dagger\, |w_1\rangle) \\
&= (\langle v|\,(|v_1\rangle), |w\rangle^\dagger\, |w_1\rangle) \\
&= (|v\rangle^\dagger\,(|v_1\rangle), |w\rangle^\dagger\, |w_1\rangle) \\
&= (|v_1\rangle, (|v\rangle^\dagger)^\dagger\, |w\rangle^\dagger\, |w_1\rangle) \\
&= (|v_1\rangle, |v\rangle\langle w|(|w_1\rangle))
\end{aligned}$$

Therefore, $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.

[**Exc**] Show that $(\sum_i \alpha_i A_i)^\dagger = \sum_i \alpha^* A_i^\dagger$.

[**Self-adjoint or Hermitian**] A linear operator $A : V \to V$ is self-adjoint or Hermitian if

$$A^\dagger = A$$

[**Exc**] The outer product $|v\rangle\langle v|$ is self-adjoint.

[**Projector**] Let V be an inner product space of dimension m and W be a subspace of V of dimension n. Let $\{|1\rangle, |2\rangle, \ldots, |n\rangle\}$ be an orthonormal basis of W and $\{|1\rangle, \ldots, |n\rangle, |n+1\rangle, \ldots, |m\rangle\}$ be an orthonormal basis of V. Let $P = \sum_{i=1}^n |i\rangle\langle i|$. Then P is an orthogonal projector on the subspace W.

[**Unitary Operator**] An operator A is called unitary if $A^\dagger A = I$.

[**Exc**] If A is an unitary operator, then $AA^\dagger = I$

[**Exc**] Unitary operators preserve inner products, i.e., if U is an unitary operator, then, for all $|v\rangle, |w\rangle$, we have

$$(|v\rangle, |w\rangle) = (U\,|v\rangle, U\,|w\rangle)$$

Indeed, $(U\,|v\rangle, U\,|w\rangle) = (U^\dagger U\,|v\rangle, |w\rangle) = (I\,|v\rangle, |w\rangle) = (|v\rangle, |w\rangle)$.

[**Outer product representation of any unitary** U] Let $\{|v_i\rangle\}$ be any orthonormal basis set. Define

$|w_i\rangle = U|v_i\rangle$, so $\{|w_i\rangle\}$ is also an orthonormal basis set. Then

$$\begin{aligned} U &= \sum_{i,j} \langle w_j|U|v_i\rangle |w_j\rangle\langle v_i| \\ &= \sum_{i,j} (|w_j\rangle, U|v_i\rangle)|w_j\rangle\langle v_i| \\ &= \sum_{i,j} (U|v_j\rangle, U|v_i\rangle)|w_j\rangle\langle v_i| \\ &= \sum_{i,j} (U^\dagger U|v_j\rangle, |v_i\rangle)|w_j\rangle\langle v_i| \\ &= \sum_{i,j} (|v_j\rangle, |v_i\rangle)|w_j\rangle\langle v_i| \\ &= \sum_{i} |w_i\rangle\langle v_i| \end{aligned}$$

[**Orthogonal Decomposition**] A orthogonal decomposition for an operator $A : V \to V$ is a representation

$$A = \sum_i \lambda_i |i\rangle\langle i|,$$

where $\{|i\rangle\}$ form an orthonormal set of eigenvectors for A, with corresponding eigenvalues λ_i.

3 Lecture#3 (Jan 27)

The Postulates of Quantum Mechanics

Quantum mechanics provides a mathematical and conceptual framework for the development of laws a physical system must obey. In the following we give a description of the basic postulates of quantum mechanics.

Postulate 1(State Space)

- Associated to any isolated physical system is an inner product space V over $\mathbb{C}$ known as the *state space* of the system. The system is completely described by its *state vector* $|\psi\rangle \in V$, which is a unit vector, i.e., $\langle\psi|\psi\rangle = 1$.

Postulate 2 (Evolution)

- The evolution of a closed quantum system is described by a *unitary operator*. Specifically, if the system is in the state $|\psi\rangle$ at time t_1, then its state $|\psi'\rangle$ at time t_2 is given by

  $$|\psi'\rangle = U_{t_1,t_2}|\psi\rangle,$$

  where U_{t_1,t_2} is a unitary operator that depends only on the times t_1 and t_2.

Postulate 3 (Quantum Measurement)

- A quantum measurement is described by a collection of measurement operators $\{M_m\}_{m=1}^n$ acting on the state space of the system. These operators satisfy the *completeness relation*

  $$\sum_{m=1}^n M_m^\dagger M_m = I$$

If the system is in the state $|\psi\rangle$ prior to measurement, then the measurement produces a classical outcome m with probability

$$\mathbb{P}[m] = \langle\psi|M_m^\dagger M_m|\psi\rangle.$$

Conditioned on obtaining classical outcome m, the post-measurement quantum state of the system is

$$|\psi_m\rangle = \frac{M_m |\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}.$$

Thus, the measurement can be viewed as a random experiment whose outcome is the pair

$$(m, |\psi_m\rangle),$$

where m represents the classical measurement outcome and $|\psi_m\rangle$ is the corresponding post-measurement quantum state.

[**Exc**] Show that $\sum_{m=1}^{n} \mathbb{P}[m] = a$

Proof:

$$\begin{aligned}
1 &= (|\psi\rangle, |\psi\rangle) \\
&= (|\psi\rangle, I|\psi\rangle) \\
&= (|\psi\rangle, (\sum_m M_m^\dagger M_m)(|\psi\rangle)) \\
&= \sum_m (|\psi\rangle, M_m^\dagger M_m(|\psi\rangle)) \\
&= \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle \\
&= \sum_m \mathbb{P}[m]
\end{aligned}$$

4 Lecture#4 (Jan 30)

[**Exc**] Consider a quantum mechanical system with state space $\mathbb{C}^2$ (with $\{|0\rangle, |1\rangle\}$ as basis) and the current state vector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$. Suppose we attempt to measure the state using the following collection of measurement operators given by the outer products $\{M_0 = |0\rangle\langle 0|, M_1 = |1\rangle\langle 1|\}$ satisfying completeness property. As per the Postulate #3, the possible set of post-measurement outputs are:

$(0, |\psi_0\rangle = \frac{M_0\psi}{\sqrt{\langle\psi|M_0^\dagger M_0|\psi\rangle}})$ with probability $\langle\psi|M_0^\dagger M_0|\psi\rangle$, and $(1, |\psi_1\rangle = \frac{M_1\psi}{\sqrt{\langle\psi|M_1^\dagger M_1|\psi\rangle}})$ with probability $\langle\psi|M_1^\dagger M_1|\psi\rangle$.

Check that

$$\begin{aligned}
\langle\psi|M_0^\dagger M_0|\psi\rangle &= (|\psi\rangle, M_0^\dagger M_0|\psi\rangle) \\
&= (M_0|\psi\rangle, M_0|\psi\rangle) \\
&= ((|0\rangle\langle 0|)(\alpha|0\rangle + \beta|1\rangle), (|0\rangle\langle 0|)(\alpha|0\rangle + \beta|1\rangle)) \\
&= (\alpha(|0\rangle\langle 0|)(|0\rangle) + \beta(|0\rangle\langle 0|)(|1\rangle), \alpha(|0\rangle\langle 0|)(|0\rangle) + \beta(|0\rangle\langle 0|)(|1\rangle)) \\
&= (\alpha|0\rangle\langle 0|0\rangle + \beta|0\rangle\langle 0|1\rangle, \alpha|0\rangle\langle 0|0\rangle + \beta|0\rangle\langle 0|1\rangle) \\
&= (\alpha|0\rangle, \alpha|0\rangle) \\
&= \alpha^*\alpha(|0\rangle, |0\rangle) \\
&= |\alpha|^2
\end{aligned}$$

Similarly, $\langle\psi|M_1^\dagger M_1|\psi\rangle = |\beta|^2$. Therefore, $|\psi_0\rangle = \frac{M_0\psi}{\sqrt{\langle\psi|M_0^\dagger M_0|\psi\rangle}} = \frac{\alpha|0\rangle}{|\alpha|}$ and $|\psi_1\rangle = \frac{M_1\psi}{\sqrt{\langle\psi|M_1^\dagger M_1|\psi\rangle}} = \frac{\beta|0\rangle}{|\beta|}$

Postulate 4 (Composite Systems)

Suppose a composite quantum system made up of two (or more) distinct physical systems. The state space of this composite system is the tensor product spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state $|\psi\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

Tensor Products

Let V, W be vector spaces over $\mathbb{C}$.

- For every $|v\rangle \in V$ and $|w\rangle \in W$, introduce a formal symbol $|v\rangle \otimes |w\rangle$. Let F be the set of all finite formal linear combinations

$$F = \left\{ \sum_{k=1}^{r} \alpha_k |v_k\rangle \otimes |w_k\rangle \ \mid r \in \mathbb{N}, \alpha_k \in \mathbb{C}, |v_k\rangle \in V, |w_k\rangle \in W \right\}$$

  Addition and scalar multiplication are defined term wise, so F is a vector space (the *free vector space* generated by the symbols $|v\rangle \otimes |w\rangle$). Note that the zero element in F is given by $\sum_k 0(|v_k\rangle \otimes |w_k\rangle)$

- Let $R \subseteq F$ be the subspace generated by the following elements: for every $|v\rangle, |v_1\rangle, |v_2\rangle \in V$, $|w\rangle, |w_1\rangle, |w_2\rangle \in W$, and $\alpha \in \mathbb{C}$

$$\begin{gathered} \alpha(|v\rangle \otimes |w\rangle) - (\alpha |v\rangle) \otimes |w\rangle, \\ \alpha(|v\rangle \otimes |w\rangle) - |v\rangle \otimes (\alpha |w\rangle), \\ (|v_1\rangle + |v_2\rangle) \otimes |w\rangle - (|v_1\rangle \otimes |w\rangle) + (|v_2\rangle \otimes |w\rangle), \\ |v\rangle \otimes (|w_1\rangle + |w_2\rangle) - (|v\rangle \otimes |w_1\rangle) + (|v\rangle \otimes |w_2\rangle) \end{gathered}$$

Finally, the tensor product is defined as the quotient vector space

$$V \otimes W = F/R$$

[**Fact**] Let $\{|i\rangle\}_{i=1}^m$ and $\{|j\rangle\}_{j=1}^n$ be the bases of V and W respectively. The basis of $V \otimes W$ is given by $\{|i\rangle \otimes |j\rangle\}_{1\le i\le m, 1\le j\le n}$. Therefore dimension of $V \otimes W$ is mn.

Brief Sketch: Take any $|v\rangle \otimes |w\rangle$. Suppose $|v\rangle = \sum \alpha_i |i\rangle$ and $|w\rangle = \sum \beta_j |j\rangle$. Then

$$\begin{aligned} |v\rangle \otimes |w\rangle &= \Big(\sum \alpha_i |i\rangle) \otimes (\sum \beta_j |j\rangle\Big) \\ &= \sum_i (\alpha_i |i\rangle) \otimes \Big(\sum \beta_j |j\rangle\Big) \\ &= \sum_{i,j} (\alpha_i |i\rangle) \otimes (\beta_j |j\rangle) \\ &= \sum_{i,j} \alpha_i \left(|i\rangle \otimes (\beta_j |j\rangle)\right) \\ &= \sum_{i,j} \alpha_i \beta_j \left(|i\rangle \otimes |j\rangle\right) \end{aligned}$$

[**Exc**] If V and W are inner product spaces over $\mathbb{C}$, then $V \otimes W$ is also an inner product space, where the inner product is given by

$$\left(\sum_i \alpha_i |v_i\rangle \otimes |w_i\rangle, \sum_j \beta_j |v_i'\rangle \otimes |w_j'\rangle \right) = \sum_{i,j} \alpha_i^* \beta_j (|v_i\rangle, |v_j'\rangle)(|w_i\rangle, |w_j'\rangle)$$

[**Exc**] Consider the vector space $\mathbb{C}^2$ with orthonormal basis $\{|0\rangle, |1\rangle\}$. Consider the tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2$. Its basis is given by

$$\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$$

We use the following natural labels for the basis elements:

$$\begin{aligned}
|00\rangle &= |0\rangle \otimes |0\rangle \\
|01\rangle &= |0\rangle \otimes |1\rangle \\
|10\rangle &= |1\rangle \otimes |0\rangle \\
|11\rangle &= |1\rangle \otimes |1\rangle
\end{aligned}$$

[**Exc**] Not every element of the tensor product space $V \otimes W$ can be written as a simple tensor of the form $|v\rangle \otimes |w\rangle$, with $|v\rangle \in V$ and $|w\rangle \in W$.

Consider the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ and the vector $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. We claim that this vector cannot be expressed as $|v\rangle \otimes |w\rangle$, for any $|v\rangle, |w\rangle \in \mathbb{C}^2$. ISuppose, for the sake of contradiction, that $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = |v\rangle \otimes |w\rangle$ where $|v\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ and $|w\rangle = \beta_0|0\rangle + \beta_1|1\rangle$. Then

$$\begin{aligned}
\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle &= (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle) \\
&= \alpha_0\beta_0|0\rangle \otimes |0\rangle + \alpha_0\beta_1|0\rangle \otimes |1\rangle + \alpha_1\beta_0|1\rangle \otimes |0\rangle + \alpha_1\beta_1|1\rangle \otimes |1\rangle \\
&= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle
\end{aligned}$$

This implies, $\alpha_0\beta_0 = 1$, $\alpha_0\beta_1 = 0$, $\alpha_1\beta_0 = 0$, and $\alpha_1\beta_1 = 1$. These equations cannot be satisfied simultaneously. Hence, $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ can not be written as a simple tensor $|v\rangle \otimes |w\rangle$.

[**Tensor of Linear Operators**]

Let V and W be two vector space with bases $\{|i\rangle\}$ and $\{|j\rangle\}$ respectively. Take any linear operators $A : V \to V'$ and $B : W \to W'$. The tensor product of A and B, denoted as $A \otimes B$, is a linear map defined as follows: for any basis element $|i\rangle \otimes |j\rangle$, define $(A \otimes B)(|i\rangle \otimes |j\rangle) = A|i\rangle \otimes B|j\rangle$, and extend linearly, i.e, for any $\sum \alpha_{i,j}|i\rangle \otimes |j\rangle$, $(A \otimes B)(\sum \alpha_{i,j}|i\rangle \otimes |j\rangle) = \sum \alpha_{i,j}(A \otimes B)(|i\rangle \otimes |j\rangle)$.

[**Exc**] Show that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

Take any $|v\rangle \in V, |w\rangle \in W, |v'\rangle \in V', |w'\rangle \in W'$, we have

$$\begin{aligned}
((A \otimes B)(|v\rangle \otimes |w\rangle), |v'\rangle \otimes |w'\rangle) &= (A|v\rangle \otimes B|w\rangle, |v'\rangle \otimes |w'\rangle) \\
&= (A|v\rangle, |v'\rangle)(B|w\rangle, |w'\rangle) \\
&= \big(|v\rangle, A^\dagger|v'\rangle\big)(|w\rangle, B^\dagger|w'\rangle) \\
&= \big(|v\rangle \otimes |w\rangle, A^\dagger|v'\rangle \otimes B^\dagger|w'\rangle\big) \\
&= \big(|v\rangle \otimes |w\rangle, (A^\dagger \otimes B^\dagger)(|v'\rangle \otimes |w'\rangle)\big)
\end{aligned}$$

Therefore, $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.